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On the Set of Obtainable Reference Trajectories Using Minimum Variance Control

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I. Introduction

In economics often the question is posed whether it is possible to track (exactly) a given set of economic target variables along any time path by means of an appropriate choice of the policy instruments. This subject is known as the target path controllability (TPC) problem. For time-invariant systems this problem has been studied in Aoki et al. (1975, 1979), Brockett et al. (1965), Maybeck (1982), Preston et al. (1972, 1974, 1982) and Tinbergen (1951), for continuous time-varying systems in Wohltmann (1985), and for discrete time-varying systems in Engwerda (1987). In this paper the TPC-problem is viewed as a problem of optimal stabilization (see Aoki (1973), Preston (1972) and Turnovsky (1973)). Only such reference time paths are called admissible which are asymptotically stabilizable, i. e. reachable in the end, by means of a control algorithm that results from the minimization of a special cost criterion.

We shall trace the set of admissible reference time paths for the minimum variance (MV) cost criterion. This criterion expresses that positive and negative deviations of target variables from desired levels are weighted equally and that they are increasingly costly. The major reasons to concentrate on this criterion are:

i) Because of the uncertainty in the real-life macro-economic situation there is a constant need for short period adaptation of control with respect to new information. A regulator which is based on minimizing a cost criterion with a short planning horizon makes such an adaptation possible.

The MV-controller is a typical example of a regulator which satisfies this requirement (see Aalders et al. (1983) and Åstrom et al. (1983, 1984).

ii) The computational ease and relatively simple formulas of the control algorithm.

However, in practice policy instruments cannot be subjected to arbitrarily large variation. This has its impact on the admissible reference trajectory set. This set will become smaller. In order to analyze the effect of limited control possibilities they are modelled in two ways.

First we alter the cost criterion. An additional term is introduced in the MV-cost criterion which penalizes quadratically a deviation of the applied control from its reference value. The advantage of this formalization is that it leads to analytical solution.

The second, more natural way to model restrictions is to assume that every control may vary only within a certain predescribed interval. However, the disadvantage of this approach is that the control scheme becomes nonlinear, and that it is impossible to give such a nice characterization of the set of admissible reference trajectories as in the firstmentioned approach. Therefore, we only state for this case a necessary and sufficient condition under which the admissibility of a reference trajectory is not affected.

To help the reader fully understand the most important theoretical results obtained in this paper, we illustrate these using a small economic model.

The description of the model itself is postponed to the last section of the paper.

II. Definitions, Tools and Mathematical Preliminaries

The base system analyzed in this paper is described by the following linear, finite dimensional, time-varying, difference equation:

$$y(k+1) = A(k)\,y(k) + B(k)\,u(k) + C(k)\;x(k); \quad k = 0, 1, \ldots \quad (1)$$

where $y(k)$ is an n-dimensional state vector observed at time k which must be controlled; $u(k)$ is an m-dimensional instruments/

control vector with $m \leq n$; $x(k)$ is a p-dimensional deterministic (non-controllable) exogenous vector which is assumed to be known at time k. The initial values of the system are $y(0)=\bar{y}(0)$ and $x(0)=\bar{x}(0)$. It is assumed that all matrices are bounded in time and that the matrices $B(k)$ are all full column rank (injective), or in other words that all the m instruments are mutually independent.

Now return to the base system (1).

Assume that a cost criterion J is given which has to be minimized.

Then, the difference between state and reference vector $y(k)-y^*(k)$, when the optimal control (w.r.t. J) is applied to the system, is called the closed-loop (CL) or control error and denoted by $e(k)$. We assume, moreover, that the number of reference/target variables always equals the number of states.

Definition

A reference trajectory is called admissible with respect to J and $y(0)$ if there exists a control function $u(\cdot)$, minimizing J, such that the corresponding error function $e(\cdot)$ converges to zero when k tends to infinity.

Note that this definition implies in particular that an admissible trajectory $y^*(k)$ has the property that it is asymptotically stabilizable.

Remark

From now on we shall omit J and $y(0)$ when we talk about the admissibility of a reference trajectory if it is clear which cost criterion and initial state are meant.

In this section we determine the set of admissible reference trajectories for the following cost criterion:

$$J=e^T(k+1)\,Q(k)\,e(k+1)+(u(k)-u^*(k))^T R(k)\,(u(k)-u^*(k)) \tag{2}$$

where $u(k)$ is the desired level of control, $Q(k)$ is a positive definite matrix, and $R(k)$ is a semi-positive definite matrix.

The reason for studying this cost criterion is that both the MV-cost criterion and its restricted version result easily from this one.

Before we state the result we introduce some notation.

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A^T will denote the transpose of matrix A, $K(A)$ an injective matrix such that $\operatorname{Im} K(A) = \operatorname{Ker} A$, and

$$M(k)(Q, R) := I - B(k)(R + B^T(k)\, Q B(k))^{-1} B^T(k)\, Q.$$

Note that then $\operatorname{Im} K \oplus \operatorname{Im} A^T = \mathbb{R}^n$ (see Lancaster et al. (1985), pp. 199).

Lemma 1:

Minimization of J subject to system (1) results in the following CL-error equation:

$$e(k+1) = M(k)\, A(k)\, e(k) + M(k)\{A(k)\, y^*(k) + B(k)\, u^*(k) + C(k)\, x(k) - y^*(k+1)\}$$

here $M(k) = M(k)(Q(k), R(k))$.

Moreover, if a reference trajectory $y^*(k)$ is admissible then there exist vector sequences $u(\cdot)$ and $v(\cdot)$, where $M(k)\, M^T(k)\, v(k)$ converges to zero when k tends to infinity, such that:

$$y^*(k+1) = A(k)\, y^*(k) + B(k)\, u^*(k) + C(k)\, x(k) + K(M(k))\, u(k) + M^T(k)\, v(k).$$

Proof:

See appendix II. □

III. The Admissible Reference Trajectories

In this section we characterize the admissible reference trajectories if MV-control is used for the regulation of the system (1).

Formally, the MV-cost criterion is obtained by taking $R(k)$ and $u^*(k)$ equal to zero in (2). As a consequence $K(M(k))$ then equals $B(k)$.

Theorem 1:

A reference trajectory is admissible w. r. t. MV-control if and only if there exist $u(\cdot)$ and $v(\cdot)$ such that the following two conditions are met:

i) $\bar{e}(0) := y(0) - y^*(0)$ is zero stabilizable by means of $v(\cdot)$ in the following linear system:

$$e(k+1) = M(k)\, A(k)\, e(k) - M(k)\, M^T(k)\, v(k); \; e(0) = \bar{e}(0).$$

That is $v(\cdot)$ is such that $e(k) \to 0$ in this equation, when $k \to \infty$.

ii) $y^*(k+1) = A(k)\, y^*(k) + B(k)\, u(k) + M^T(k)\, v(k)$.

Proof:

"$\Rightarrow$" Since $K(M(k)) = B(k)$, we obtain immediately from lemma 1 that condition ii) is always satisfied if $y^*(\cdot)$ is admissible.

Substitution of ii) into the corresponding control error equation (see lemma 1 again) yields that:

$$e(k+1) = M(k)\, A(k)\, e(k) - M(k)\, M^T(k)\, v(k),$$
$$\text{where } M(k)\, M^T(k)\, v(k) \to 0 \text{ when } k \to \infty$$
$$e(0) = \bar{e}(0).$$

So, we have that $\bar{e}(0)$ is stabilized, which completes this part of the proof.

"$\Leftarrow$" Equation ii) implies that $M(k)\{A(k)\, y^*(k) + C(k)\, x(k) - y^*(k+1)\}$ equals $-M(k)\, M^T(k)\, v(k)$ at any time k.

So, the error equation can be rewritten as.

$$e(k+1) = M(k)\, A(k)\, e(k) - M(k)\, M^T(k)\, v(k)$$
$$e(0) = \bar{e}(0).$$

By assumption i), this vector sequence $v(\cdot)$ also stabilizes $\bar{e}(0)$. Consequently, a reference trajectory satisfying assumption i) and ii) will be admissible. □

Theorem 1 leads to the, intuitively very appealing, result that any admissible reference trajectory must satisfy a recurrence equation which corresponds to the system. The only difference with the system is that the reference trajectory may possess an additional disturbation, which converges to zero when k tends to infinity. Note that this necessary condition is also sufficient if the norm of the matrix $M(k)\, A(k)$ is, at any time k, smaller than one. For, in that case, condition i) of theorem 1 is trivially satisfied.

So, policy makers cannot assign arbitrarily desired values to their target variables. In general these policy objectives will conflict. If they stick to these variables, consequently, the only way left open to realize their goals is to pursue a policy that is aimed at changing the structural (system) parameters. Moreover, we see that the policy makers of small countries (like the Netherlands) in their choice of admissible reference trajectories depend severely on what happens in foreign countries.

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0002 Another consequence of this theorem is that a rank condition for the solvability of the TPC problem can be easily obtained.

0003 *Corollary 1:*

0004 Every path $y^*(\cdot)$ will be an admissible reference path for system (1) if and only if the number of instruments is greater than or equal to the number of target variables.

0005 *Proof:*

0006 From theorem 1 it is clear that in case the number of instruments is equal to the number of states, every reference path may be admissible.

0007 Straightforward calculation shows that matrix $M(k)$ equals zero.

0008 Therefore, we conclude from the error equation that the control error equals zero for all k. So, any reference path is admissible in this case.

0009 On the other hand, when the number of instruments is smaller than the number of states, it is easy to construct a reference trajectory which is not admissible. This completes the proof. □

0010 A last remark w.r.t. the theorem concerns the influence of the weighting matrix Q on the admissibility of a reference trajectory.

0011 From the second condition of theorem 1 it is clear that this matrix has no direct influence on the admissibility. However, indirectly it has its impact because it does influence the stability of the CL-error equation. Engwerda and Otter studied this aspect of the MV-controller in more detail in (1986).

0012 IV. Time-Invariant Systems

0013 In this section we take a closer look at the existence condition of theorem 1, if the system matrices $A(k)$, $B(k)$ and $C(k)$ are constant in time.

0014 To that end we introduce first some well-known concepts and notation. For formal definitions and proofs we refer the reader to standard textbooks like Kailath (1980), Kwakernaak et al. (1972) and Wonham (1974).

0015 Consider the system $y(k+1)=Ay(k)+Bu(k)$; $y(0)=\bar{y}(0)$. Let **R** be the set of states reachable from $\bar{y}(0)=0$. Then **R** equals the linear subspace $\operatorname{Im} B+A\operatorname{Im} B+\cdots+A^{n-1}\operatorname{Im} B$. This subspace is

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usually abbreviated by $\langle A \mid B \rangle$. It can then be proved that the set of all initial states that can be controlled to zero consists of $\{y(0) \mid A^{n-1} y(0) \in \langle A \mid B \rangle\}$. We will abbreviate this set by $\langle A \mid B \rangle_0$.

Furthermore, the set of initial states $y(0)$ for which the solution of $y(k+1) = Ay(k)$ converges to zero when k tends to infinity is called the stable subspace and is denoted by $X^-(A)$.

The following proposition for time-invariant systems is well-known.

Proposition 1:

$y(0)$ is zero stabilizable (i. e. there exists a control function $u(\cdot)$ such that $y(k) \to 0$, when $k \to \infty$) if and only if $y(0)$ belongs to the subspace $X^-(A) + \langle A \mid B \rangle_0$.

Now let matrix M be as defined in section II.

Then theorem 1 passes into:

Theorem 1':

A reference trajectory is admissible w.r.t. MV-control if and only if there exist $u(\cdot)$ and $v(\cdot)$ such that the following two conditions are met.

i) $v(\cdot)$ is such that $e(\cdot) \to 0$ in:

$$e(k+1) = MA\,e(k) - MM^T v(k); \quad e(0) = y(0) - y^*(0).$$

ii) $y^*(k+1) = Ay^*(k) + Bu(k) + Cx(k) + M^T v(k)$. □

From proposition 1 it is now immediately clear that w.r.t. theorem 1' the next corollary holds.

Corollary 2:

There exists an $e(0)$ stabilizing vector sequence $v(\cdot)$ if and only if $e(0)$ is an element of the subspace $X^-(MA) + \langle MA \mid MM^T \rangle_0$. □

For the Kendrick model (see section VI), where $m = n = 2$, the condition of corollary 2 is trivially satisfied since $M = 0$ and therefore $X^-(MA) = \mathbb{R}^2$ and $\langle MA \mid MM^T \rangle_0$. A closer analysis shows that in this case any trajectory is admissible. In fact this is due to Tinbergen's rank condition, i. e. rank $B = n$ (see also corollary 1).

V. The Influence of Control Restrictions

In this section the influence of control restrictions on the admissibility of a reference trajectory is investigated.

First, we analyze the effect of costly controls.

We assume that the cost criterion is given by (2), in which $R(k)$ now is taken positive definite. As a result, $M(k)(Q(k), R(k))$ is then positive definite too, which implies that $K(M(k))$ is zero. These considerations lead to theorem 2. Its proof is analogous to that of theorem 1.

Theorem 2:

A reference trajectory is admissible if and only if there exists a vector sequence $v(\cdot)$ such that the following two conditions are met:

i) $\bar{e}(0) := y(0) - y^*(0)$, is zero stabilizable by means of $v(\cdot)$ in the following linear system:

$$e(k+1) = M(k)\, A(k)\, e(k) - M(k)\, M^T(k)\, v(k); \quad e(0) = \bar{e}(0).$$

ii) $$y^*(k+1) = A(k)\, y^*(k) + B(k)\, u^*(k) + C(k)\, x(k) + {} + M^T(k)\, v(k). \qquad \square$$

Note that the requirement that the applied control should not deviate too much from its set point trajectory, $u^*(k)$, shows up nicely in the admissibility condition. In contrast with section III, where an admissible reference trajectory which was disrupted by any control sequence remained admissible, we see now that only those reference trajectories from which the dynamic evaluation is in correspondence with the set point trajectory $u^*(k)$ are admissible.

So the more policy-makers are pinned to control reference trajectories, the less freedom they have in the choice of an admissible target trajectory. Or, taking another point of view, the more policy-makers stick to desired control trajectories the more the influence of the non-controllable exogenous variables on the admissible target paths will be. For, in that case one cannot react accurately on any unexpected change in these variables.

For time-invariant systems the same property w.r.t. the existence of an $\bar{e}(0)$ stabilizing sequence $v(\cdot)$ as in corollary 2 holds. In section VII we use this result in the Kendrick example.

Now assume that the restrictions are modelled as follows:

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$$\underset{u(k)}{\text{minimize}}\, e^T(k+1)\, Q(k)\, e(k+1), \text{ given } \underline{d}_i(k) \le u_i(k) \le \bar{d}_i(k)$$

where $u(k) = (u_1(k), \cdots, U_m(k))$ and $\underline{d}_i(k)$ and $\bar{d}_i(k)$ are given constants $i = 1, \ldots, m$.

Then, if the control possibilities are restricted for l consecutive time steps, the effect on the admissibility of a reference trajectory can be characterized exactly.

Before the theorem concerning this subject is discussed, we introduce some notation.

From now on $u^a(k)$ will denote the applied control at time step k and $u^{\text{opt}}(k)$ the optimal control at time step k if there would exist no bounds on the permitted extent of control as this time step. Furthermore, will $\Delta u(k)$ denote the difference betwen $u^a(k)$ and $u^{\text{opt}}(k)$.

Theorem 3:

If the control possibilities are restricted for l consecutive time steps, then a reference trajectory that is admissible in the sense of section III remains admissible if and only if $B\Delta u(k_0+l-1) + MAB\Delta u(k_0+l-2) + \ldots + (MA)^{l-1} B\Delta u(k_0)$ belongs to the stable space of matrix MA.

Here k_0 denotes the time at which the bounds first became effective.

Proof:

See appendix II. □

VI. The Influence of White Noise

In this section we shall briefly comment on what happens with the admissibility conditions when the system (1) is disturbed by white noise. In fact we will show that these conditions remain the same. To prove this, we shall reconsider some formulas. The problem to be solved is now to minimize the expected quadratic cost functional $E\{J\}$, see section II, subject to the constraint:

$$y(k+1) = A(k)\,y(k) + B(k)\,u(k) + C(k)\,x(k) + w(k),$$

where $w(k)$ is a white-noise bector with cov. $\{w(k)\, w^T(l)\} = \Sigma_w\, \delta_{kl}$. Here δ_{kl}, the kronecker delta, equals 1 if $k = l$ and 0 otherwise.

Due to the fact that the system noise is white, we know that the optimal control minimizing $E\{J\}$ is equal to the optimal control obtained in the proof of lemma 1 (see appendix II). So the CL-error equation now becomes:

$$e(k+1) = M(k)\,A(k)\,e(k) + M(k)\,\{A(k)\,y^*(k) + B(k)\,u^*(k) + C(k)\,x(k) - y^*(k+1)\} + w(k).$$

From this we obtain that the expected CL-error is given by the recurrence equation:

$$E\{e(k+1)\} = M(k)\,A(k)\,E\{e(k)\} + M(k)\,\{A(k)\,y^*(k) + B(k)\,u^*(k) + C(k)\,x(k) - y^*(k+1)\}.$$

So, the results of theorems 1 and 2 remain valid, in the sense that they give necessary and sufficient conditions for convergence of the expected CL-error to zero.

However, in case the closed-loop system matrix MA is not asymptotically stable the error covariance grows to infinity. So, in general for practical situations these theorems make no sense. To have a more valuable analogue of theorem 1 for time-invariant systems, we state the following corollary:

Corollary 3:

Let MA be stable and let there exist $u(\cdot)$ and $v(\cdot)$ such that $y^*(k+1) = Ay^*(k) + Bu^*(k) + Cx(k) + K(M)\,u(k) + M^T v(k)$, where $v(\cdot) \to 0$.

Then, $y^*(\cdot)$ is admissible (in the sense that the expected CL-error converges to zero) and the error covariance remains bounded. □

The other results mentioned in the previous sections can be generalized in the same way. We do not go into any further detail about this subject.

VII. A Simulation Study

The simulation study is performed on a macro-economic model estimated by Kendrick for the U.S. economy in (1982).

He considered the following reduced form model:

$$\begin{bmatrix} C(k) \\ I(k) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} C(k-1) \\ I(k-1) \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} u_1(k-1) \\ u_2(k-1) \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} x(k) + \begin{bmatrix} v_1(k) \\ v_2(k) \end{bmatrix}$$

where $C(k)$ = Private Consumption;
$I(k)$ = Gross Private Investment;
$u_1(k)$ = Governmental Expenditures;
$u_2(k)$ = Money Supply;
$x(k)$ = (Non-Controllable) Exogenous variable;
$V^T(k) = (v_1^T(k)\, v_2^T(k))$ is a white noise vector with $\operatorname{cov}\{V(k)\, V^T(s)\} = \Sigma_v \delta_{ks}$.

All quantities are measured in billions of dolllars, in quarter k.

The parameters obtained by Kendrick are:

$$A = \begin{bmatrix} 0.914 & -0.016 \\ 0.097 & 0.424 \end{bmatrix}; \; B = \begin{bmatrix} 0.305 & 0.424 \\ -0.101 & 1.459 \end{bmatrix};$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} x(k) = \begin{bmatrix} -59.437 \\ -184.766 \end{bmatrix}$$

$$\Sigma_v = \begin{bmatrix} 3.73 & 0 \\ 0 & 8.58 \end{bmatrix}, \text{ with initial values } C(0) = 387.9 \text{ and } I(0) = 85.3\,.$$

The reference trajectories are given by the following recurrence equations:

$$\begin{bmatrix} C^*(k) \\ I^*(k) \end{bmatrix} = \begin{bmatrix} 1.0075 & 0 \\ 0 & 1.0075 \end{bmatrix} \begin{bmatrix} C^*(k-1) \\ I^*(k-1) \end{bmatrix}$$

and

$$\begin{bmatrix} u_1^*(k) \\ u_2^*(k) \end{bmatrix} = \begin{bmatrix} 1.0075 & 0 \\ 0 & 1.0075 \end{bmatrix} \begin{bmatrix} u_1^*(k-1) \\ u_2^*(k-1) \end{bmatrix}$$

with initial values $C^*(0) = 387.9$; $I^*(0) = 85.3$; $u_1^*(0) = 110.4$ and $u_2^*(0) = 157.3$. The considered weighting matrices used in the numerical exercise are:

$$Q = \begin{bmatrix} 0.0625 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } R = \begin{bmatrix} 1 & 0 \\ 0 & 0.44 \end{bmatrix}.$$

Note that the reference trajectories are defined as growth paths, and that matrix B is invertible.

From corollary 1 we know that in case the number of instruments equals the number of states any reference trajectory is admissible when MV-control is used to regulate the system. This is illustrated in Fig. 1. In this experiment we set R equal to zero, that is, we applied MV-control. We see that indeed the consumption- and investment reference trajectory are exactly tracked. When

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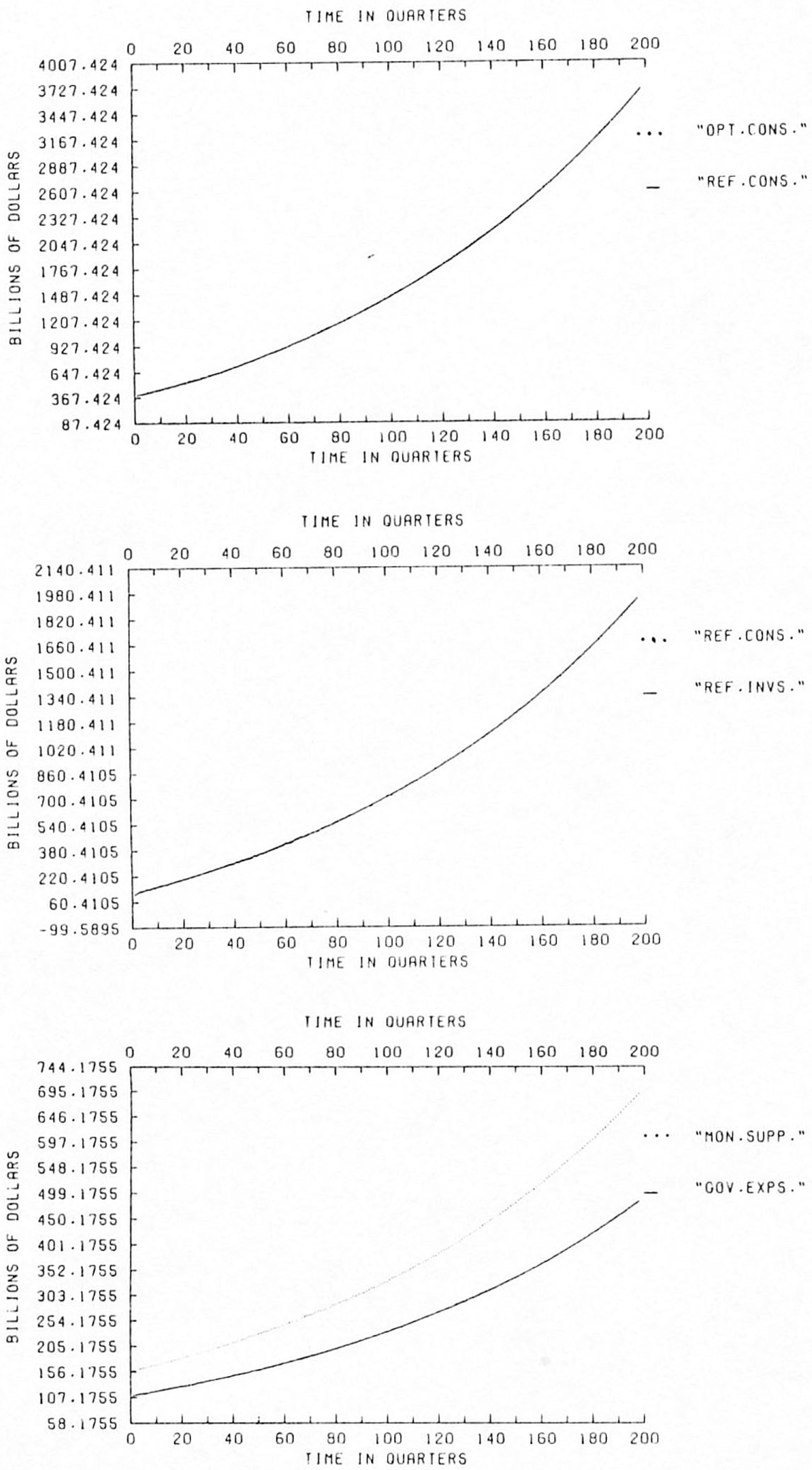

Fig. 1

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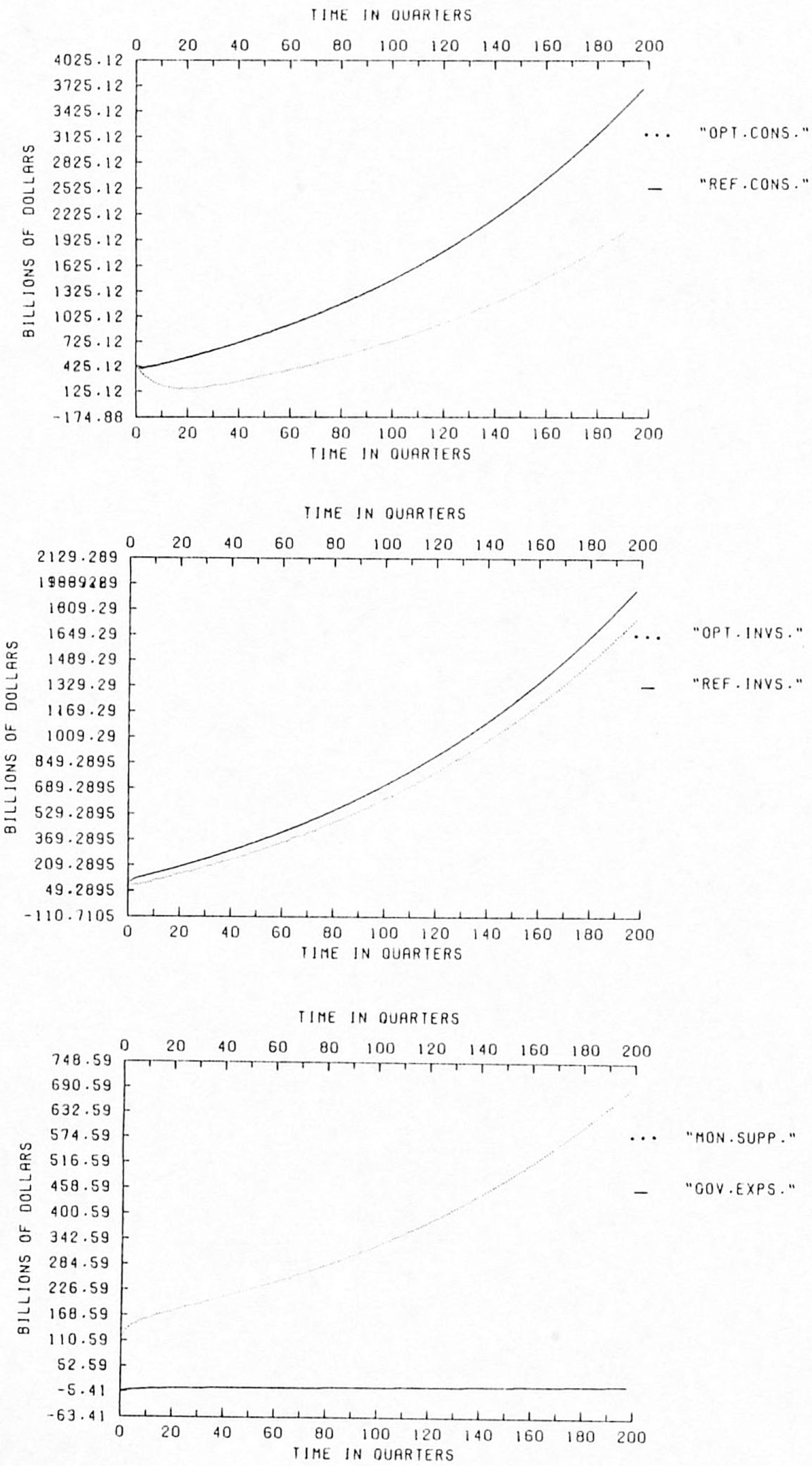

Fig. 2

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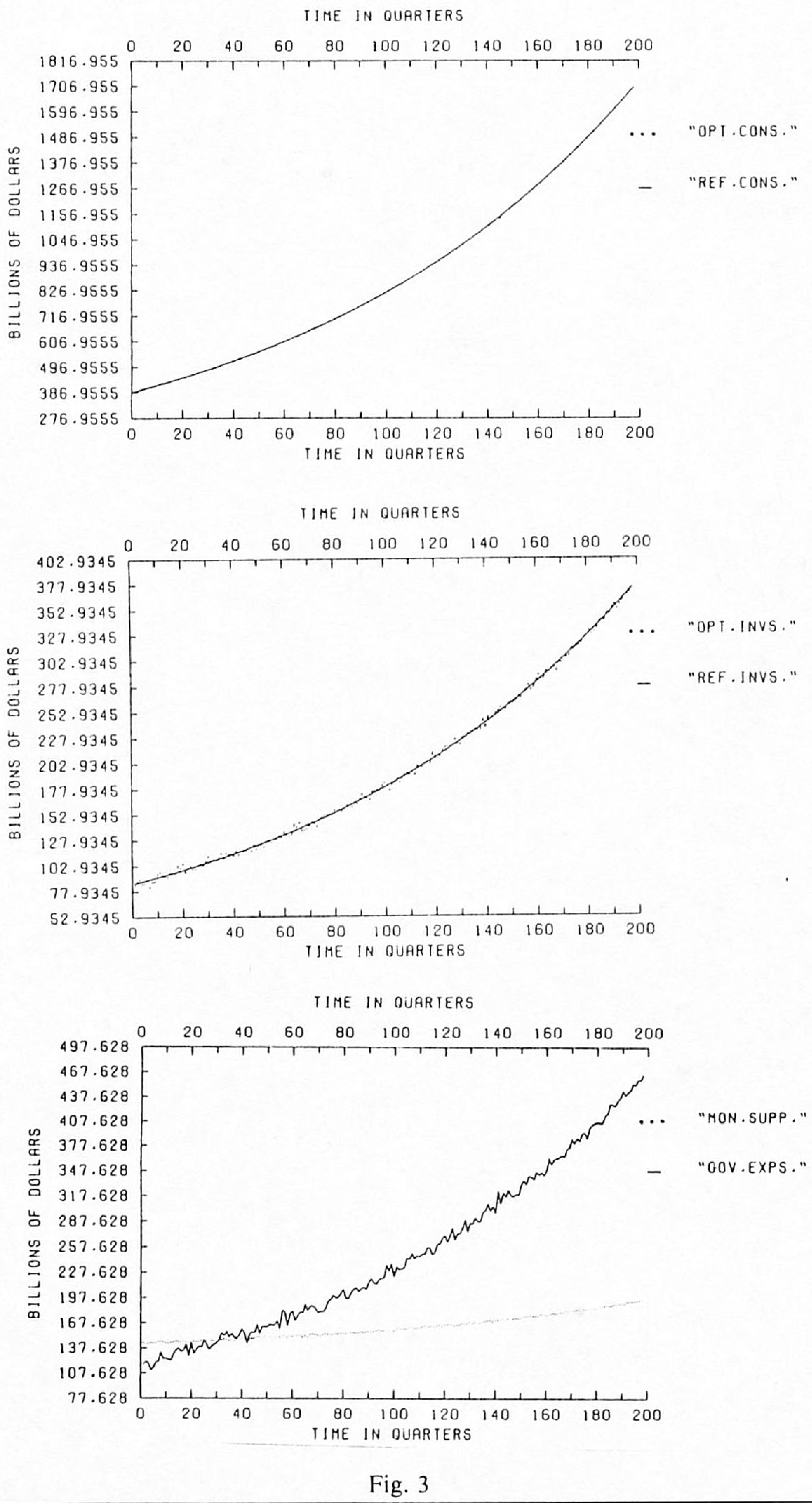

Fig. 3

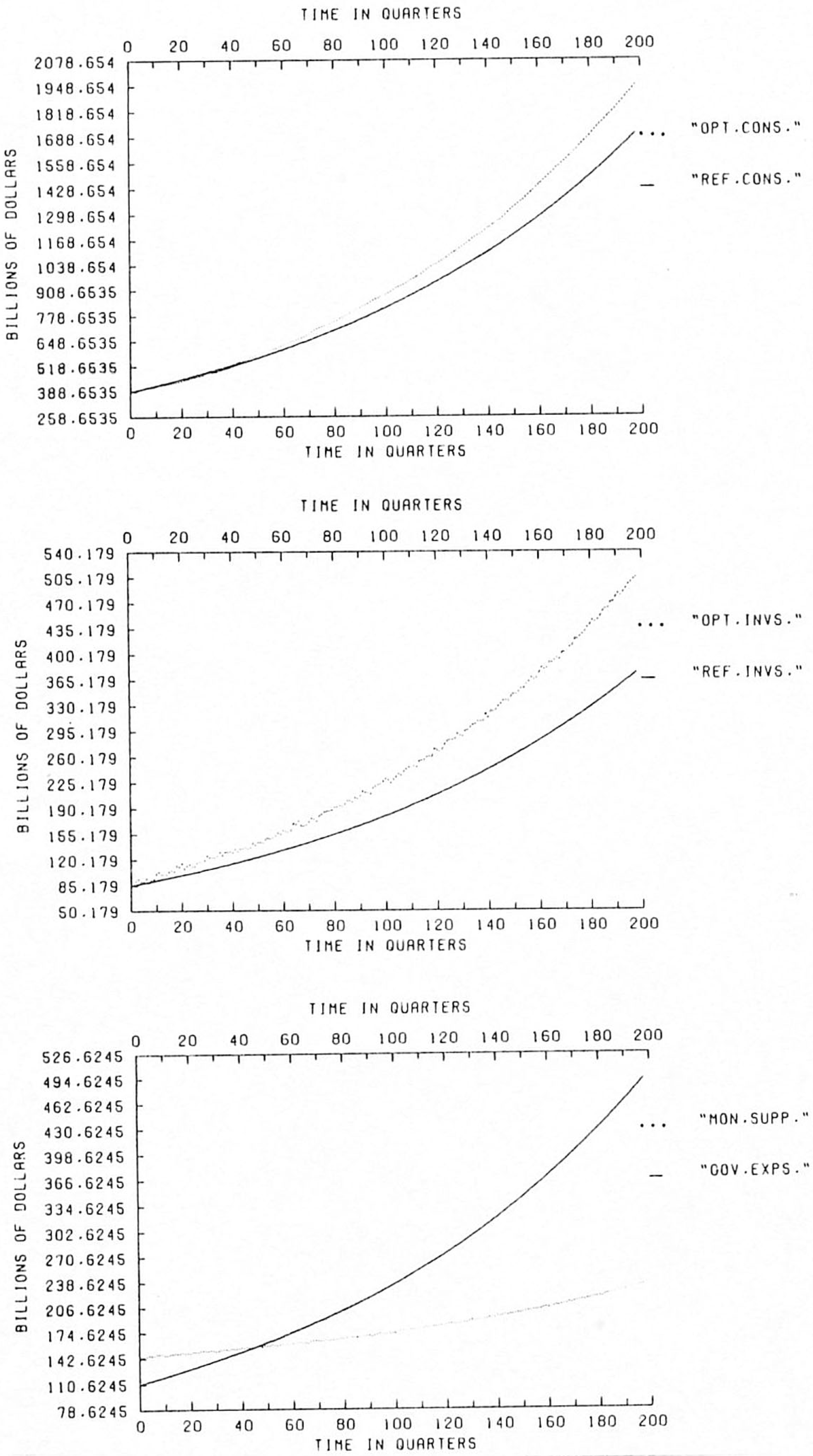
TIME IN QUARTERS
BILLIONS OF DOLLARS
"OPT.CONS."
"REF.CONS."
"OPT.INVS."
"REF.INVS."
"MON.SUPP."
"GOV.EXPS."

Fig. 4

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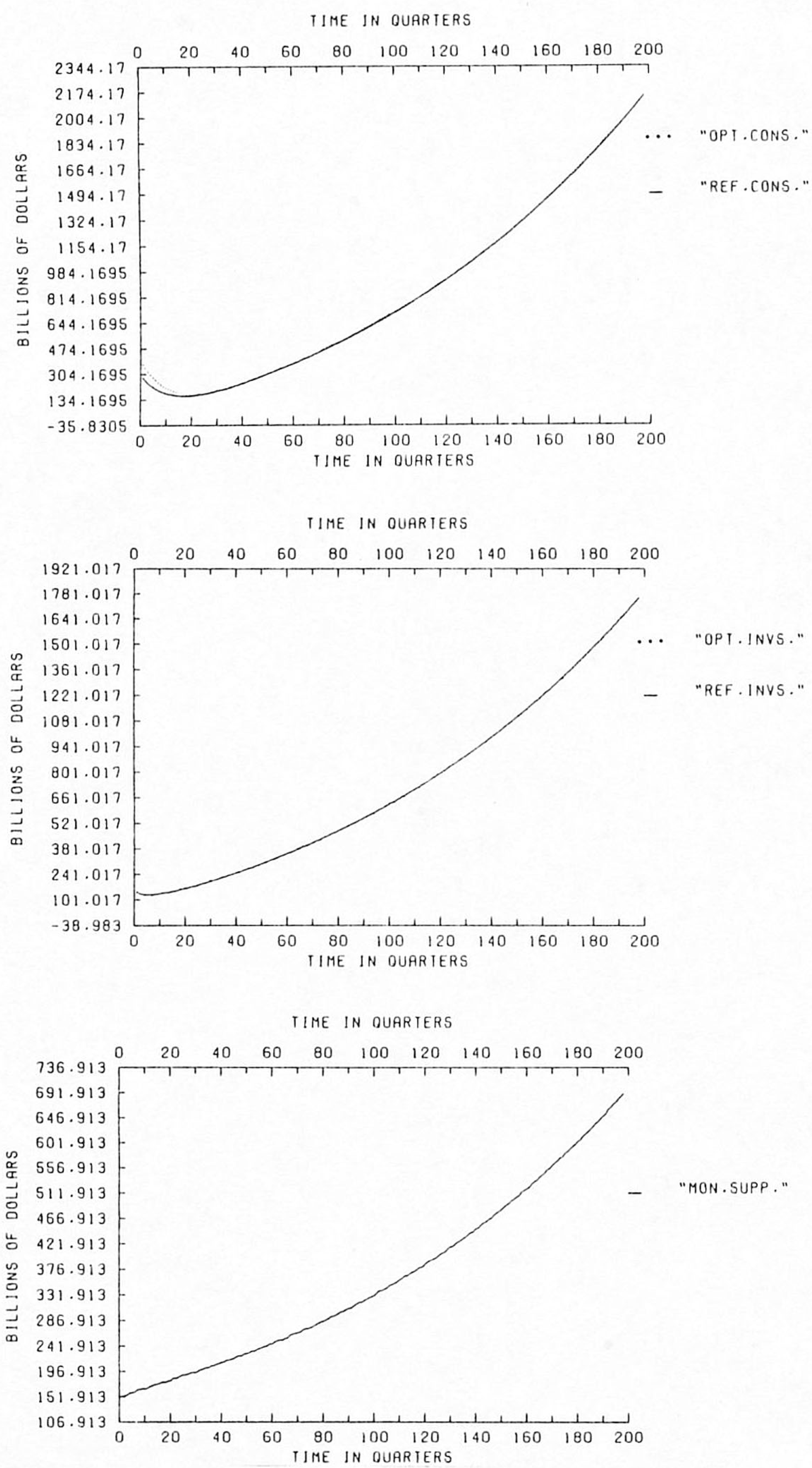

Fig. 5

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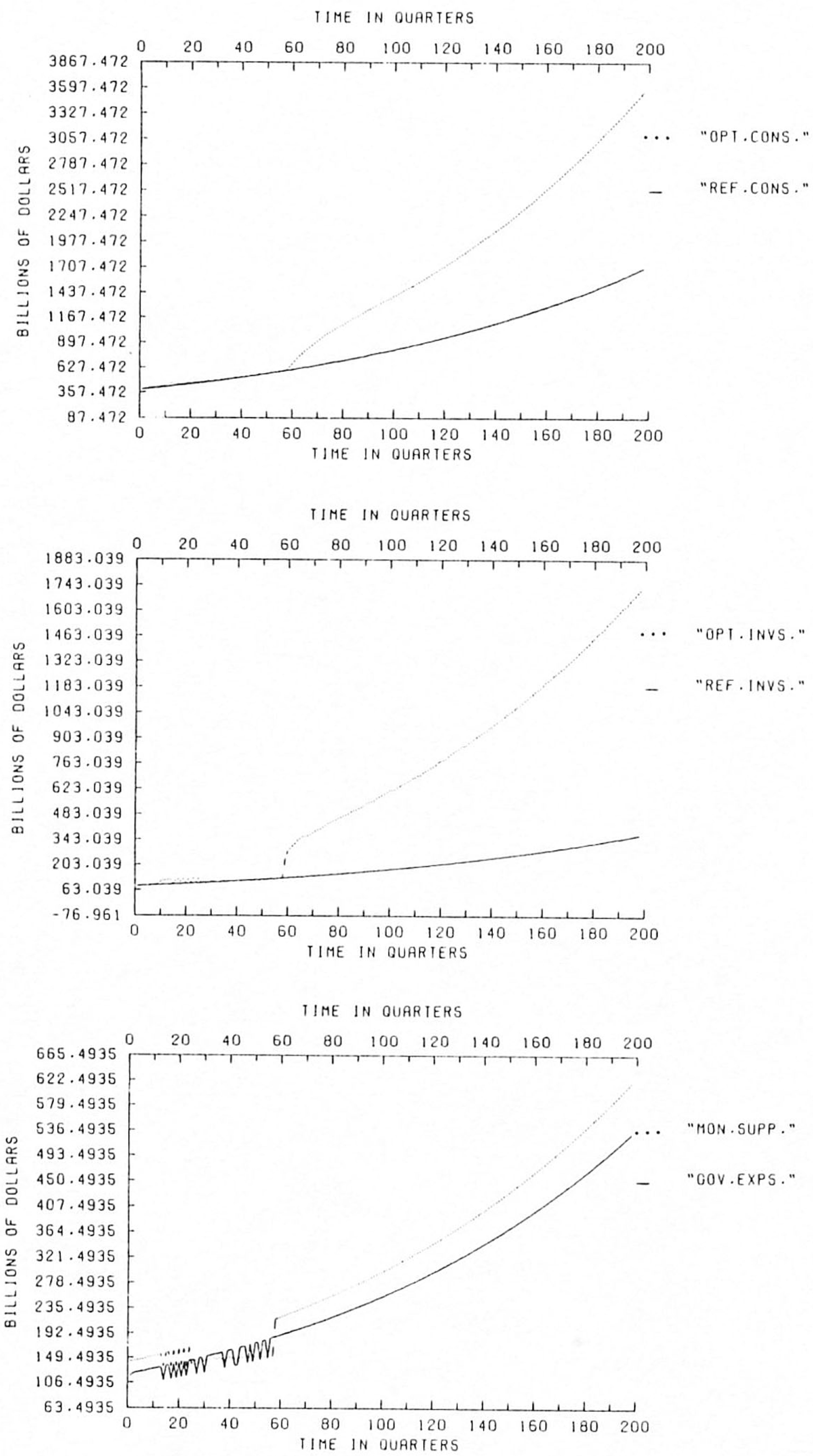

Fig. 6

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control costs are taken into account we know from section V that things change drastically.

Straightforward calculation shows that in this model the eigenvalues of MA are $0.582 \pm 0.195\, i$. So, they are situated inside the unit circle which implies that $X^{-}(MA) = \mathbb{R}^2$ again. Note that also $\langle MA \mid MM^T \rangle_0$ equals $\mathbb{R}^2$. Therefore a reference trajectory is admissible in this case if and only if it is generate as follows:

$$y^*(k+1) = Ay^*(k) + Bu^*(k) + Cx(k) + v(k);$$
$$y(0) = \bar{y}(0) \text{ and } v(\cdot) \to 0.$$

In Fig. 2 the simulation results are given for the model described above. We observe that the obtained closed-loop system is now unstable, though its system matrix MA is asymptotically stable. By altering the consumption- and investment reference trajectory conform theorem 2 we see that the system becomes stable again (Fig. 3). It is essential in this case that in the generation of these trajectories not only the desired trajectory for the control variables is considered. This is shown in Fig. 4. Here we see that when the chosen exogenous variable sequence used in the generated consumption- and investment reference trajectory differs from the real one, the system becomes unstable again. The trajectory of the exogenous variable in this experiment was taken identically zero, and the desired control trajectories similar to experiment 3.

In order to show that in case the number of instruments is smaller than the number of controlled variables there are still a lot of consumption- and investment trajectories that can be tracked, an experiment is performed with one control and two states. The chosen instrument is the money supply, and MV-control is applied to regulate the system. The model parameters are taken as above. As a result the closed-loop system matrix MA is asymptotically stable again. To satisfy the conditions of theorem 1 the reference trajectories of consumption and investment are generated with a growth matrix equal to A, an arbitrarily desired control trajectory and the exogenous variable path of the system. The initial reference values for consumption and investment were respectively 300 and 170. The results are shown in Fig. 5.

At last the effect of bounding the control absolute is simulated (see section V). In this experiment we assume that the control reference trajectory is generated like in experiment 2. The permitted deviation of the applied control from this setpoint trajectory is assumed to be at any time at most ten percent. The

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simulation results are shown in Fig. 6. From Fig. 6iii we see that the money supply brings on the instability of the closed-loop system. Furthermore we see that during the first sixty quarters the tracking properties are somewhat better than in experiment 2 and that the control exhibits a bang-bang behaviour. When both control bounds become effective we see that the destabilization effects are much greater than those in experiment 2.

This in spite of the fact that the total amount of control applied to the system is greater (as well for the money supply as for the government expenditures).

Conclusions

In this paper we showed that any MV-admissible reference trajectory must satisfy a recurrence equation which corresponds to the system. In general this dynamic evolution condition is not enough to conclude admissibility. An additional necessary and sufficient condition for admissibility is that components of noise appearing outside the image of matrix B and which do not show up in the system are stabilized by the closed-loop system matrix.

When control costs are introduced in the MV-cost criterion, this affects directly the admissibility conditions.

Any admissible reference trajectory now explicitly takes acount of the dynamics of the desired control variables.

From the simulations we see that if hard bounds for the controls in the MV-cost criterion are introduced (see second part section V) this results in the short run to a better tracking of the target trajectories. In that case, however, the control exhibits a bang-bang behaviour, a property which is undesirable form a practical point of view. Moreover the consequences of this policy are disastrous in the long run. Compared with the control cost formulation of the problem we see that more control effort is needed to obtain a much worse tracking result. So as well from a mathematical point of view as from a practical point of view, analyzing limited control possibilities by introducing costly controls seems to be preferable.

An indirectly reobtained result in this paper is Tinbergen's counting rule for the TPC-problem. We showed that a discrete time-varying system, with matrix B injective, is TPC if and only if the number of instruments is greater or equal than the number of states.

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Appendix I

Appendix II

Proof of Lemma 1:

By straightforward differentiation of cost criterion J it is seen that the optimal control for system (1) minimizing J is:

$$u(k) = -(B^T(k)\,Q(k)\,B(k) + R(k))^{-1}\,B^T(k)\,Q(k)\,(A(k)\,y(k) + C(k)\,x(k) - y^*(k+1)) + (B^T(k)\,Q(k)\,B(k) + R(k))^{-1}\,R(k)\,u^*(k).$$

Substitution of this control into base system (1) leads to the following control error equation:

$$e(k+1) = M(k)\,A(k)\,y(k) + M(k)\,C(k)\,x(k) - M(k)\,y^*(k+1) + B(k)\,(B^T(k)\,Q(k)\,B(k) + R(k))^{-1}\,R(k)\,u^*(k),$$

where $M(k) = M(k)\,(Q(k), R(k))$ is as defined in section II.

Some matrix manipulation shows that

$$B(k)\,(B^T(k)\,Q(k)\,B(k) + R(k))^{-1}\,R(k)\,u^*(k)$$
$$\text{equals } M(k)\,B(k)\,u^*(k).$$

So the above error equation can be rewritten as:

$$e(k+1) = M(k)\,A(k)\,e(k) + M(k)\,(A(k)\,y^*(k) + C(k)\,x(k) + B(k)\,u^*(k) - y^*(k+1)),$$

which proves the first part of the lemma.

Now, since $M(k)\,A(k)$ is bounded, a necessary condition for

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convergence of $e(k)$ to zero is that $M(k)(A(k)\,y^*(k) + B(k)\,u^*(k) + C(k)\,x\,(k) - y^*(k+1))$ converges to zero.

Since the kernel of $M(k)$ is in general unequal to zero, we can rewrite this last condition as:

$$y^*(k+1) = A(k)\,y^*(k) + B(k)\,u^*(k) + C(k)\,x(k) + \\ + M^T(k)\,v(k) + K(M(k))\,u(k)$$

for some $u(k)$ and $v(k)$, where $M(k)\,M^T(k)\,v(k)$ converges to zero when k tends to infinity. □

Proof of Theorem 3:

Let $e^{(q)}(k_0+k)$ denote the control error at time k_0+k, when from time k_0 to k_0+q application of optimal control is not possible.

Then, it follows from the identity $y(i+1) - y^*(i+1 = A\,y(i) + B\,u(i) + Cx(k) - y^*(i+1)$ that the equations

$$e^{(i)}(k_0+i) = A\,y(k_0+i-1) + B\,u^a(k_0+i-1) + \\ + Cx(k_0+i-1) - y^*(k_0+i)$$

and

$$e^{(i-1)}(k_0+i) = A\,y(k_0+i-1) + B\,u^{\text{opt}}(k_0+i-1) + \\ + Cx(k_0+i-1) - y^*(k_0+i) \text{ hold.}$$

Subtracting one equation from the other then yields:

(a) $$e^{(i)}(k_0+i) = B(u^a(k_0+i-1) - u^{\text{opt}}(k_0+i-1)) + \\ + e^{(i-1)}(k_0+i).$$

Note that the admissible trajectories with respect to the cost criterion considered in this subsection are a subset of those considered in section III.

So, when after $i-1$ time steps the bounds on the extent of control disappear, we have from theorem 2 that $e^{(i-1)}(k_0+i)$ is equal to

(b) $$M A\,e^{(i-1)}(k_0+i-1) - M M^T v(k_0+i).$$

Substitution now of (b) into (a) gives.

$$e^{(l)}(k_0+l) = B\Delta u(k_0+l-1) + MAB\Delta u)k_0+l-2) + \ldots \\ + (MA)^{l-1} B\Delta u(k_0) + e(k_0+l),$$

where $e(k_0+l) = e^{(0)}(k_0+l)$.

Since no restrictions exist anymore from time step l on, it is

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easily shown by induction that for k greater than zero.

$$e^{(l)}(k_0 + l + k) - e(k_0 + l + k)$$

equals $MA\,(e^{(l)}(k_0 + l + k - 1) - e(k_0 + l + k - 1))$.

The last two equations imply the result as stated in the theorem. □

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